Interest Calculations

In the worked examples here, the following formulas will be used: (note: it applies to compound interest)

- The future value (A) of an initial amount (P) (initial amount can also be referred to as "principal") over n periods (periods can be months, years, quarters, etc.) at an interest rate i, is calculated by

$$A = P(1+i)^n \qquad (1)$$

- The future value (F) of payments, each payment x paid at the end of each period over n periods, where the interest rate is i , is calculated by

$$F = \frac{x\left[(1+i)^n - 1\right]}{i} \qquad (2)$$

- The present value (P) of payments that are to be paid in the future, each payment x being paid at the end of each period over n periods, where the interest rate is i , is calculated by

$$P = \frac{x\left[1-(1+i)^{-n}\right]}{i} \qquad (3)$$

Example 1

A loan of €1 000 000 is taken out at an interest rate of 9% p.a. compounded monthly. The loan is to be repaid over a period of 20 years by monthly payments. Calculate:

a. The monthly payment

b. The total amount paid at the end of the 20-year period

c. The total interest paid at the end of the 20-year period

d. The amount still owing after 5 years

a. The monthly payments are to be such that their present value is equal to the initial amount of the loan. To determine the monthly payment, formula (3) will be used and x (the monthly payment) will be made the subject of the formula.

P = €1 000 000, $i = \frac{0.09}{12}$ (the interest rate is put as decimal and then divided by 12 as there is monthly compounding) and n = 240 months (20 years × 12 months in year).

$$x=\frac{Pi}{\left[1-(1+i)^{-n}\right]}$$

$$=\frac{1000000\times\frac{0.09}{12}}{\left[1-\left(1+\frac{0.09}{12}\right)^{-240}\right]}$$

$$=€8\ 997.26$$

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b. The total amount paid will simply be

€8997.26 per month × 240 months = €2 159 342.40

c. The total interest paid will be the total amount paid at the end of the period less the principal at the start, ie, €2 159 342.40 – €1 000 000 = €1 159 342.40

d. What this question is referring to is the balance of principal still due. Each monthly payment comprises some principal and some interest so as time goes by, the amount of principal still owing decreases. The amount of principal owing at a certain point in time is the present value of the instalments that will still be paid throughout the remaining time of the loan. To calculate this value, formula (3) will be applied where *x* is the monthly payment of €8 997.26, *i* = $\frac{0.09}{12}$ and *n* will be the time that remains. In this case, since 5 years have gone by and the time of the loan is 20 years, 15 years remain so *n* = 180 months (15 years × 12 months). So, the amount still owing can be calculated:

$$P=\frac{x\left[1-(1+i)^{-n}\right]}{i}$$

$$=\frac{8997.26\left[1-\left(1+\frac{0.09}{12}\right)^{-180}\right]}{\frac{0.09}{12}}$$

$$=€887\ 070.49$$

Example 2

A loan is taken out over a period of 10 years at an interest rate of 8% p.a. compounded quarterly. The loan is repaid in instalments of €9 138.95, paid at the end of every quarter. Calculate the principal amount of the loan (to the nearest euro).

The principal amount will be the amount initially borrowed and will therefore be the present value of all the instalments that will be paid throughout the time of the loan. So, formula (3) will be applied.

Quarterly compounding means that the compounding takes place every quarter of a year (ie., every 3 months). The instalments are also paid every 3 months. Since compounding takes place 4 times a year, $i = \frac{0.08}{4}$ and n = 40 (10 years × 4 times a year). x = 9 138.95

So,

$$P = \frac{x\left[1-(1+i)^{-n}\right]}{i}$$

$$= \frac{9138.95\left[1-\left(1+\frac{0.08}{4}\right)^{-40}\right]}{\frac{0.08}{4}}$$

=€250 000.36

=€250 000 (rounded)

Example 3

Refer back to the loan in Example 1. Suppose that after 8 years, the interest rate is raised by 50 basis points. Calculate:

a. The new monthly payments that will apply

b. The total amount paid during the time of the loan

c. The total interest paid during the time of the loan

d. The diffence in interest paid due to the rate being changed

a. Step 1. The outstanding principal balance after 8 years (96 months) has to be determined. This will be the principal used to determine the new monthly payments for the remaining 12 years (144 months). This will be the present value of the old payments under the old interest rate were they to be applied for the remaining time of the loan.

So, formula (3) will be applied with x = €8 997.26, $i = \frac{0.09}{12}$, and n = 144 months.

This will give

$$P = \frac{x\left[1-(1+i)^{-n}\right]}{i}$$

$$= \frac{8997.26\left[1-\left(1+\frac{0.09}{12}\right)^{-144}\right]}{\frac{0.09}{12}}$$

=€790 599.06

Step 2. Apply this principal balance to to the new interest rate and the remaining time of the loan and use formula (3) to get the new monthly payment. One basis point is 0.01%, which means that the interest rate was raised by 0.5%, making the new interest rate 9.5% p.a. So, *i* = $\frac{0.095}{12}$. *n* = 144 months.

$$x = \frac{Pi}{\left[1-(1+i)^{-n}\right]}$$

$$= \frac{790599.06 \times \frac{0.095}{12}}{\left[1-\left(1+\frac{0.095}{12}\right)^{-240}\right]}$$

=€9 221.34

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b. The total amount paid throughout the time of the loan will be the total of all the monthly payments for the first 8 years (96 months) under the old interest rate plus the total of all the monthly payments for the remaining 12 years (144 months).

For the first 8 years, the total will be €8 997.26 × 96 = €863 736.96.

For the remaining 12 years, the total will be €9 221.34 × 144 = €1 327 872.96

So, the total payments throughout the time of the loan will be

€863 736.96 + €1 327 872.96 = <u>€2 191 609.92</u>

c. Total interest paid is total amount paid less principal.

This will be €2 191 609.92 – €1 000 000 = €1 191 609.92

d. Were the rate to have remained unchanged, the total interest paid would have been €1 159 342.40. Due to the rate being changed, it became €1 191 609.92.

The difference is thus €1 191 609.92 - €1 159 342.40 = €32 267.52. So, €32 267.52 more has been paid in interest due to the rate change.

Example 4

A certain girl decides she wants to have saved €1 000 000 over a 10-year period by depositing a fixed amount every month in an investment that offers an interest rate of 6% p.a. compounded monthly. She decides to start at the end of that very month. Calculate the monthly amount she is to deposit in order to fulfil her desire.

In order for this to happen, the future value of all these amounts that she has deposited over the 10-year period is to be €1 000 000. So, formula (2) will be used with *x* (the monthly amount) being the subject of the formula. F = €1 000 000, $i = \frac{0.06}{12}$ $n = 120$ months.

$$x=\frac{Fi}{\left[(1+i)^{n}-1\right]}$$

$$=\frac{1000000\times\frac{0.06}{12}}{\left[\left(1+\frac{0.06}{12}\right)^{120}-1\right]}$$

=€6 102.05