The exponential function

$f(x) = a^x \qquad a > 0.$

Find $f'(x)$:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$
$$= \lim_{h\to 0} \frac{a^{x+h} - a^x}{h}$$
$$= \lim_{h\to 0} \frac{a^x . a^h - a^x}{h}$$
$$= \lim_{h\to 0} \frac{a^x (a^h - 1)}{h}$$
$$= a^x \lim_{h\to 0} \frac{a^h - 1}{h}.$$

For any $a > 0$, $\lim_{h\to 0} \frac{a^h - 1}{h}$ does actually exist and it is denoted by $k(a)$, i.e. $k(a) = \lim_{h\to 0} \frac{a^h - 1}{h}$

So, $f'(x) = k(a)\, a^x$ $\left(\text{or } \frac{d}{dx} a^x = k(a) a^x \right)$

Now, let's determine $k(a)$ for various values of a:

a	1	2	3
$k(a)$	0	0,666134	1,110223

Somewhere between 2 and 3 it must occur that $k(a) = 1$.
This happens at the number 2,7182818.........

It's an irrational number and is called e

$k(e) = 1 \quad \lim_{h \to 0} \frac{e^h - 1}{h} = 1$

So, if $f(x) = e^x$, $f'(x) = e^x$. $\left(\text{or } \frac{d}{dx} e^x = e^x\right)$

$f''(x) = e^x$, $f^{(3)}(x) = e^x$, ..., $f^{(n)}(x) = e^x$.

Since $\frac{d}{dx} e^x = e^x$, this means on the graph of e^x, at any point on the graph we have the situation that the gradient is the y-value there.

If c is any real number, $\frac{d}{dx} ce^x = ce^x$.

$$\frac{d^2}{dx^2} ce^x = ce^x$$

$$\frac{d^3}{dx^3} ce^x = ce^x$$

$$\vdots$$

$$\frac{d^n}{dx^n} ce^x = ce^x.$$

In the graph of ce^x, we also have the situation that the gradient at any point on the graph is the y-value there.

$$\lim_{x \to 0} \frac{e^x - 1}{x} = 1.$$